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TRANSIENT ANALYSIS OF A THREE-DIMENSIONAL PLATE BY THE RAY GROUPING TECHNIQUE†

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Abstract—The propagation of transient waves in an elastic plate excited by an arbitrary loading is investigated. Exact three-dimensional transform solutions are derived for a plate suddenly loaded on one of its bounding surfaces, with the inversion effected by Cagniard's technique. The solution is based on three canonical problems for finding the n-th reflected waves from only the information on the $(n-1)$-th reflected waves. The technique automatically groups rays which arrive simultaneously at one point, thus simplifying the computations needed by the ray tracing technique. A nonaxially symmetric sample loading is selected for demonstrating the technique, and numerical results are presented. The Ray Grouping Technique can be extended to the layered medium case.

INTRODUCTION

The propagation of transient waves in an elastic thick plate excited by an arbitrary loading is considered here for a load which is suddenly applied on one of its bounding surfaces. The disturbance is analyzed by using one-sided and two-sided Laplace transforms, and the inversion scheme is based on the well-known Cagniard technique. The transform solution for these loads is given in the form of an infinite series which is based on three canonical problems. A mode conversion matrix is formed in each problem. The technique consists of finding the n-th reflected and transmitted wave from only the information on the $(n-1)$-th reflected and transmitted wave. The interpretation of the infinite series, which represents multi-reflected shear and dilatational waves, is verified by a "Ray Grouping Technique". This technique formulates the solution in transform space, and simplifies the calculation in spite of the generality of the loading function. Using the mode conversion matrix, the physical meaning of each term of the infinite series is discussed, and a proof of the convergence of the infinite series is given.

A nonaxially symmetric sample loading is chosen for application of the technique, and only a finite number of terms is required to determine the velocities for any particular finite time of interest. The final expressions are used to deduce the Green's function for the velocity, and numerical results are presented for a step function input. The three-dimensional sample problem cannot be treated by computer codes which have been set up for either two-dimensional problems or axially symmetric problems[1, 3]; however, the problem appears to be within the capabilities of Threedy[23].

In geophysical applications, the elastic layered medium is the most common model used in the analytical investigation of Earth motion. For this model, most analytic solutions[4–9] have been restricted to simple loadings, such as point, infinite line and axially symmetric loads, and very few results[10, 11] have been obtained for finite loads. Since wave propagation in a plate and in a layered medium are analogous[4, p. 281], the present solution may serve as a guide for problems involving layered half-spaces acted upon by finite loads.

Previous analytic work on the transient solution of the elastic plate[12–15] has been largely confined to the plane strain condition, simple loading functions, and the axially symmetric conditions. For more general types of loading, Norwood[16] performed the theoretical analysis for loads with finite characteristic dimensions on an elastic plate with plane strain condition,

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developing also the "Ray Grouping Technique", a simple and clear-cut technique to obtain reflectivity and transmissibility coefficients. The present investigation is an extension of Norwood's work into a fully three-dimensional analysis. Here, the main objective is to find the most suitable analytical method for studying elastic wave propagation in a thick plate with arbitrary loading conditions, and the present work will be extended to the general solution in stratified media.

For the short time, the solution in an elastic plate is equivalent to that of the elastic half-space. Based on this fact, the plate problem can be separated into three canonical problems; each canonical problem is solved in transform space and written in matrix form. Each solution yields a mode conversion matrix which generates the wave forms and guides the direction of waves. The plate solution is found as a sum of solutions of these canonical problems.† In the ray tracing method[7, 8, 9], one may find, for example, a term identified by *SPSP------P*. Beginning from the left, this is a wave which started as a shear wave (S), reflected as a dilatational wave (P), reflected as a shear wave (S), and so on, and is currently a dilatational wave (P). Suppose that, disregarding the last P, P appears m times and S appears n times. Based on the ray tracing techniques, one needs to consider every possible ordering of the P and S symbols, thus yielding $(m+n)!/m!n!$ terms. Using the present techniques, all of these terms appear as only one term, and therefore the present method is called the Ray Grouping Technique. Obviously, the present development simplifies the analysis of the plate problem. Moreover, since the technique and results presented here are quite general, their extension to other problems, such as layered medium problems, may be accomplished quite readily.

PROBLEM STATEMENT AND SOLUTION TECHNIQUE

In a rectangular coordinate system, consider an elastic plate confined to $0 \le z \le h$. An arbitrary load is applied on the bounding surface $z = 0$, as shown in Fig. 1.

The governing wave equations are

$$c_1^2\nabla^2\varphi = \frac{\partial^2\varphi}{\partial t^2}, \qquad c_2^2\nabla^2\boldsymbol{\psi} = \frac{\partial^2\boldsymbol{\psi}}{\partial t^2}, \qquad \nabla\cdot\boldsymbol{\psi} = 0. \tag{1}$$

The potentials φ and $\boldsymbol{\psi}$ are related to the displacements through

$$\mathbf{U} = \nabla\varphi + \nabla x\boldsymbol{\psi} \tag{2}$$

where c_1 and c_2 are the wave speeds, $\rho c_1^2 = \lambda + 2\mu$, $\rho c_2^2 = \mu$, λ and μ are the Lamé constants,

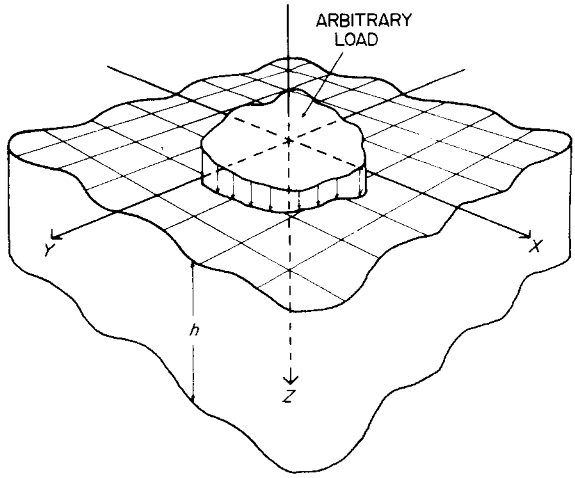

Fig. 1. Geometry of the general case.

†The basic method was proposed in [24], and was rederived almost simultaneously in [14, 16].

and ρ is the material density. The stress-strain relations for this medium are

$$\tau_{ij} = \lambda \nabla^2 \varphi \delta_{ij} + 2\mu \epsilon_{ij}, \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{3}$$

where δ_{ij} is the Kronecker delta. The initial conditions are taken as

$$\varphi(x, y, z, 0) = \frac{\partial \varphi}{\partial t}(x, y, z, 0) = \boldsymbol{\psi}(x, y, z, 0) = \frac{\partial \boldsymbol{\psi}}{\partial t}(x, y, z, 0) = 0$$

representing quiescence at $t = 0$. The boundary conditions for the problem are

$$\begin{gathered}\tau_{zz}(x, y, 0, t) = \mu g(x, y, t)H(t); \qquad \tau_{yx}(x, y, 0, t) = \mu h(x, y, t)H(t) \\ \tau_{xz}(x, y, 0, t) = \mu i(x, y, t)H(t)\end{gathered} \tag{4a}$$

where $g(x, y, t)$, $h(x, y, t)$ and $i(x, y, t)$ are loading functions assumed to be Laplace transformable in all three variables. At $z = h$, one imposes the conditions (although more general boundary conditions could be specified):

$$\tau_{xz}(x, y, h, t) = \tau_{yz}(x, y, h, t) = \tau_{zz}(x, y, h, t) = 0. \tag{4b}$$

The potentials φ, $\boldsymbol{\psi}$ and the space derivatives of the potential are required to vanish at infinity; that is, they must satisfy the radiation condition.

$$\lim_{|x|,|y| \to \infty} [\varphi(x, y, z, t), \boldsymbol{\psi}(x, y, z, t), \text{etc.}] = 0. \tag{5}$$

FORMAL SOLUTION

In view of the initial, boundary and radiation conditions, the appropriate transforms for the problem are the one-sided and two-sided Laplace transforms defined respectively by the equations

$$\bar{f}(x, y, z, p) = \mathscr{L}(f) = \int_0^\infty f(x, y, z, t)\, e^{-pt}\, dt \tag{6a}$$

$$f(x, y, z, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(x, y, z, p)\, e^{pt}\, dp \tag{6b}$$

$$\bar{f}^{LL}(k, v, z, p) = \int_{-\infty}^{\infty}\int \bar{f}(x, y, z, p)\, e^{p(kx+vy)}\, dx\, dy \tag{7a}$$

$$\bar{f}(x, y, z, p) = \frac{-p^2}{(2\pi)^2}\int_{-i\infty-\epsilon}^{i\infty-\epsilon}\int \bar{f}^{LL}(k, v, z, p)\, e^{-p(kx+vy)}\, dk\, dv \tag{7b}$$

where p, k, and v are the Laplace transform parameters, and c is chosen to the right of any singularity of $\bar{f}(x, y, z, p)$. In accordance with Lerch's theorem[18], it is sufficient to assume in eqns (6) and (7) that p is a real positive number; for this guarantees the existence of a unique inverse. In eqn (7b) "ϵ" lies within the strip of convergence[17], so that the inversion path in (7b) lies within the strip of convergence. By applying these transforms in sequence to (1), in conjunction with the initial conditions, boundary conditions (4a) and radiation condition (5), one finds the ordinary differential equations and their solutions

$$\frac{d^2\bar{\varphi}^{LL}}{dz^2} = \eta_1^2(k, v)\bar{\varphi}^{LL}; \qquad \frac{d^2\bar{\boldsymbol{\psi}}^{LL}}{dz^2} = \eta_2^2(k, v)\bar{\boldsymbol{\psi}}^{LL} \tag{8}$$

$$\bar{\varphi}^{LL}(k, v, z, p) = A(k, v, p)\, e^{-p\eta_1(k, v)z} + A^*(k, v, p)\, e^{p\eta_1(k, v)z} \tag{9a}$$

$$\bar{\boldsymbol{\psi}}^{LL}(k, v, z, p) = \mathbf{B}(k, v, p)\, e^{-p\eta_2(k, v)z} + \mathbf{B}^*(k, v, p)\, e^{p\eta_2(k, v)z} \tag{9b}$$

where $\eta_i(k, v) = (a_i^2 - k^2 - v^2)^{1/2}$; $a_i c_i = 1$, $i = 1, 2$ and $Re\, \eta_i(k, v) \geq 0$ and Re denotes the real part. From eqns (9), there exist eight unknown coefficients to be determined from the boundary conditions. To find these unknowns, one has to solve a system of eight equations by tedious and complicated algebra.

The usual procedure of transform calculus will lead to a very difficult form for the inversion process. However, in most applications, one is interested only in a few reflections; thus most of the terms in these equations provide superfluous information. Based on the suggestion by Mencher[19], and demonstrated in a plane strain problem by Norwood[16], one proceeds to adopt the Ray Grouping Technique to analyze the present problem. This technique will provide the terms required for the first few reflections in a more straightforward manner, and also provide the total solution.

RAY GROUPING TECHNIQUE

The basic idea of this technique is to obtain the reflectivity and transmissibility coefficients with a very simple procedure, and these coefficients, due to the effects of the plane interface, will provide the information to analyze a bounded medium with an interface or layered medium. The solution to the posed problem can be obtained in a wave expansion form by considering three canonical problems, where each problem contributes a mode conversion matrix which will guide the direction of rays and group them. The geometry of these canonical problems is the three-dimensional analog of Fig. 2 in [16].

Canonical problem A

For this problem, one considers an elastic half-space to which the arbitrary loading function is applied on the surface $z = 0$, as denoted by (4a), and the solutions needed from (9) must be bounded for large z. Therefore, the solution for the half-plane is obtained by setting $A^*(k, v, p)$ and $\mathbf{B}^*(k, v, p)$ in eqn (9) equal to zero, and deleting the boundary conditions at $y = h$. By defining $H(k, v, p)$, $I(k, v, p)$, and $G(k, v, p)$ as the functions $h(x, y, t)$, $i(x, y, t)$, and $g(x, y, t)$ in transform space, and $R(k, v)$ as the Rayleigh function $R(k, v) = (a_2^2 - 2k^2 - 2v^2)^2 + 4\eta_1(k, v)\eta_2(k, v) \cdot (k^2 + v^2)$, and by noting that the third of eqns (1) implies that

$$vB_y(k, v, p) = -kB_x(k, v, p) - \eta_2(k, v)B_z(k, v, p),$$

it follows that the solution for the displacement potentials may be represented in matrix form using a 3 by 3 matrix:

$$\begin{pmatrix} \tilde{\varphi}^{LL}(k, v, z, p) \\ \tilde{\psi}_x^{LL}(k, v, z, p) \\ \tilde{\psi}_z^{LL}(k, v, z, p) \end{pmatrix} = \mathcal{H}(-z)[p^2 a_2^2 \eta_2(k, v) R(k, v)]^{-1} A(k, v) J(k, v, p) \tag{10}$$

where

$$A(k, v) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & 0 \end{pmatrix}$$

$$J(k, v, p) = \begin{pmatrix} H(k, v, p) \\ I(k, v, p) \\ G(k, v, p) \end{pmatrix}$$

and

$$A_{11} = 2va_2^2\eta_2^2(k, v)$$

$$A_{12} = 2ka_2^2\eta_2^2(k, v)$$

$$A_{13} = a_2^2\eta_2(k, v)(a_2^2 - 2k^2 - 2v^2)$$

$$A_{21} = 4k^2\eta_1(k, v)\eta_2(k, v) + (a_2^2 - 2k^2)(a_2^2 - 2k^2 - 2v^2)$$

$$A_{22} = -2kv\eta_2(k, v)[2\eta_1(k, v)\eta_2(k, v) - (a_2^2 - 2k^2 - 2v^2)]$$

$$A_{23} = -2a_2^2 v\eta_1(k, v)\eta_2(k, v)$$

$$A_{31} = -kR(k, v)$$

$$A_{32} = vR(k, v)$$

$$\mathscr{H}(-z) = \begin{pmatrix} e^{-p\eta_1 z} & 0 & 0 \\ 0 & e^{-p\eta_2 z} & 0 \\ 0 & 0 & e^{-p\eta_2 z} \end{pmatrix}.$$

$\mathscr{H}(-z)$ is the mode conversion matrix introduced by Norwood[16] and extended to the three-dimensional case.

Canonical problem B

In this problem, one considers an elastic half-space bounded below by the surface $z = h$. An incident wave, travelling from the interior to the surface $z = h$, is given by the potentials

$$\begin{aligned} \tilde{\varphi}_i^{LL}(k, v, z, p) &= A(k, v, p)\, e^{-p\eta_1(k, v)z} \\ \tilde{\boldsymbol{\psi}}_i^{LL}(k, v, z, p) &= \mathbf{B}(k, v, p)\, e^{-p\eta_2(k, v)z} \end{aligned} \tag{11a}$$

where the subscript i indicates the incident wave, and $A(k, v, p)$, $\mathbf{B}(k, v, p)$ are some given functions.

For this problem, one needs to determine the potentials of the reflected field based on a zero stress boundary condition at the surface $z = h$. The reflected potentials are given by

$$\begin{aligned} \tilde{\varphi}_r^{LL}(k, v, z, p) &= C(k, v, p)\, e^{p\eta_1(k, v)z} \\ \tilde{\boldsymbol{\psi}}_r^{LL}(k, v, z, p) &= \mathbf{D}(k, v, p)\, e^{p\eta_2(k, v)z} \end{aligned} \tag{11b}$$

where $C(k, v, p)$, $\mathbf{D}(k, v, p)$ are to be determined.

The total potential field is the sum of the incoming incident potential field and the outgoing reflective potential field, and is given as

$$\begin{aligned} \tilde{\varphi}^{LL}(k, v, z, p) &= A(k, v, p)\, e^{-p\eta_1(k, v)z} + C(k, v, p)\, e^{p\eta_1(k, v)z} \\ \tilde{\boldsymbol{\psi}}^{LL}(k, v, z, p) &= \mathbf{B}(k, v, p)\, e^{-p\eta_2(k, v)z} + \mathbf{D}(k, v, p)\, e^{p\eta_2(k, v)z}. \end{aligned} \tag{12}$$

The application of the boundary conditions (4b) leads to

$$\begin{pmatrix} \tilde{\varphi}_r^{LL}(k, v, p) \\ \tilde{\psi}_{rx}^{LL}(k, v, p) \\ \tilde{\psi}_{rz}^{LL}(k, v, p) \end{pmatrix} = \mathscr{H}(z-h)\mathscr{R}_p(k, v) \begin{pmatrix} \tilde{\varphi}_i^{LL}(k, v, h, p) \\ \tilde{\psi}_{ix}^{LL}(k, v, h, p) \\ \tilde{\psi}_{iz}^{LL}(k, v, h, p) \end{pmatrix} \tag{13}$$

where

$$\mathscr{R}_p(k, v) = v^{-1}R^{-1}(k, v) \begin{pmatrix} \mathscr{R}_{11} & \mathscr{R}_{12} & \mathscr{R}_{13} \\ \mathscr{R}_{21} & \mathscr{R}_{22} & \mathscr{R}_{23} \\ 0 & 0 & \mathscr{R}_{33} \end{pmatrix}$$

$$\mathscr{R}_{11} = -vS(k, v)$$

$$\mathscr{R}_{12} = 4(k^2 + v^2)\eta_2(k, v)(a_2^2 - 2k^2 - 2v^2)$$

$$\mathscr{R}_{13} = 4k\eta_2^2(k, v)(a_2^2 - 2k^2 - 2v^2)$$

$$\mathscr{R}_{21} = -4v^2\eta_1(k, v)(a_2^2 - 2k^2 - 2v^2)$$

$$\mathscr{R}_{22} = -vS(k, v)$$

$$\mathscr{R}_{23} = 8kv\eta_1(k, v)\eta_2^2(k, v)$$

$$\mathscr{R}_{33} = vR(k, v)$$

and $\mathcal{H}(z-h)$ is the mode conversion matrix for this problem. $\mathcal{R}_p(k, v)$ represents the reflectivity matrix at the surface $z = h$, and $S(k, v) = (a_2^2 - 2k^2 - 2v^2)^2 - 4\eta_1(k, v)\eta_2(k, v)(k^2 + v^2)$.

Canonical problem C

In this problem, a half space is given by $z > b$, with the free boundary at $z = b$. A group of incident waves travelling from the interior upward to the surface is given in terms of the elastic potentials and one must find the potentials of the reflected wave. By following the analysis in the previous canonical problem, one obtains the potentials in the matrix form.

$$\begin{pmatrix} \tilde{\varphi}_r^{LL}(k, v, z, p) \\ \tilde{\psi}_{rx}^{LL}(k, v, z, p) \\ \tilde{\psi}_{rz}^{LL}(k, v, z, p) \end{pmatrix} = \mathcal{H}(b-z)\mathcal{R}_N(k, v) \begin{pmatrix} \tilde{\varphi}_i^{LL}(k, v, b, p) \\ \tilde{\psi}_{ix}^{LL}(k, v, b, p) \\ \tilde{\psi}_{iz}^{LL}(k, v, b, p) \end{pmatrix} \tag{14}$$

where

$$\mathcal{R}_N(k, v) = v^{-1}R^{-1}(k, v) \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{pmatrix}$$

$$R_{11} = -vS(k, v)$$

$$R_{12} = -4(k^2 + v^2)\eta_2(k, v)(a_2^2 - 2k^2 - 2v^2)$$

$$R_{13} = 4k\eta_2^2(k, v)(a_2^2 - 2k^2 - 2v^2)$$

$$R_{21} = 4v^2\eta_1(k, v)(a_2^2 - 2k^2 - 2v^2)$$

$$R_{22} = vS(k, v)$$

$$R_{23} = -8kv\eta_1(k, v)\eta_2^2(k,_v)$$

$$R_{33} = vR(k, v),$$

$\mathcal{R}_N(k, v)$ represents a reflection matrix at the surface, $z = b$, and $\mathcal{H}(b-z)$ is the mode conversion matrix for this problem. Note that $\mathcal{R}_N$ is the matrix inverse of $\mathcal{R}_p$.

Application of the results

In the original formulation of the problem, there were eight unknowns which could be found from the given six boundary conditions (4a) and (4b) and the divergence condition. The solution from this system of eight equations would lead to the usual form as obtained by the regular transform technique. This form is extremely difficult to invert, even in the two-dimensional axially symmetric problem[13]. Based on the present technique, there are only four unknowns to be determined in each canonical problem and they will lead to forms which can be inverted exactly, requiring at most a convolution for an arbitrary time input.

Canonical problem A serves to begin the process of finding the various terms in the solution. The results of the canonical problem A serve as the incident wave for the canonical problem B; then the results of the canonical problem B serve as the incident wave for the canonical problem C. The process is repeated by taking the results from problem C as the incident disturbance for problem B. Symbolically, this process is represented by A→B→C→B→C→B→ . . ., where each preceding canonical problem's solution serves as the incident field for the next canonical problem. By continuing this process indefinitely, the displacement potentials can be generated.

Define $TP(k, v, z, p)$ as the total displacement potential vector, and $PM((k, v, z, p); n)$ as the potential vector for the n-th reflection by

$$TP(k, v, z, p) = \begin{pmatrix} \tilde{\varphi}^{LL}(k, v, z, p) \\ \tilde{\psi}_x^{LL}(k, v, z, p) \\ \tilde{\psi}_z^{LL}(k, v, z, p) \end{pmatrix}; \qquad PM((k, v, z, p); n) = \begin{pmatrix} \tilde{\varphi}^{LL}(k, v, z, p) \\ \tilde{\psi}_x^{LL}(k, v, z, p) \\ \tilde{\psi}_z^{LL}(k, v, z, p) \end{pmatrix}_n$$

such that

$$TP(k, v, z, p) = \sum_{n=0}^{\infty} PM((k, v, z, p); n) \tag{15}$$

where the first term ($n = 1$) is obtained directly from the canonical problem A and subsequent terms are found recursively as will be indicated. Thus,

$$PM((k, v, z, p); 1) = [p^2 a_2^2 \eta_2(k, v) R(k, v)]^{-1} \mathscr{H}(-z) A(k, v) J(k, v). \tag{15a}$$

The result for $n = 1$ is now used as the incident wave for canonical problem B, and the result represents the second term of $TP(k, v, z, p)$; that is,

$$PM((k, v, z, p); 2) = [p^2 a_2^2 \eta_2(k, v) R(k, v)]^{-1} \mathscr{H}(z-h) \mathscr{R}_p(k, v) \mathscr{H}(-h) A(k, v) J(k, v, p). \tag{15b}$$

The result for $n = 2$ now serves as the incident wave for canonical problem C, where b is set equal to zero, and the result represents the third term of $TP(k, v, z, p)$; that is,

$$\begin{aligned} PM((k, v, z, p); 3) &= \mathscr{H}(0-z) \mathscr{R}_N(k, v) PM((k, v, z, p); 2) \\ &= [p^2 a_2^2 \eta_2(k, v) R(k, v)]^{-1} \mathscr{H}(-z) \mathscr{M}(k, v, h) A(k, v) J(k, v, p) \end{aligned} \tag{15c}$$

where $\mathscr{M}(k, v, h) = \mathscr{R}_N(k, v) \mathscr{H}(-h) \mathscr{R}_p(k, v) \mathscr{H}(-h) = (m_{ij})$. $\mathscr{M}(k, v, h)$ represents the 3 by 3 reflection matrix which effects the ray grouping and will be called the "Ray Grouping Matrix". The elements of this matrix are given by

$$\begin{aligned}
M(k, v) &= R^{-1}(k, v)[(a_2^2 - 2k^2 - 2v^2)^2 - 4\eta_1(k, v)\eta_2(k, v)(k^2 + v^2)], \\
m_{11} &= M^2(k, v) X^2 - [M^2(k, v) - 1] YX \\
m_{12} &= 4R^{-2}(k, v) v^{-1} \eta_2(k, v)(k^2 + v^2)(a_2^2 - 2k^2 - 2v^2) S(k, v) Y(Y - X) \\
m_{13} &= 4R^{-2}(k, v) v^{-1} k \eta_2^2(k, v)(a_2^2 - 2k^2 - 2v^2) S(k, v) Y(Y - X) \\
m_{21} &= 4R^{-2}(k, v) v \eta_1(k, v)(a_2^2 - 2k^2 - 2v^2) S(k, v) X(Y - X) \\
m_{22} &= M^2(k, v) Y^2 - [M^2(k, v) - 1] XY \\
m_{23} &= 16R^{-2}(k, v) k \eta_1(k, v) \eta_2^2(k, v)(a_2^2 - 2k^2 - 2v^2)^2 T(X - Y) \\
m_{31} &= m_{32} = 0 \\
m_{33} &= Y^2
\end{aligned}$$

with $X = e^{-p\eta_1(k, v)h}$ and $Y = e^{-p\eta_2(k, v)h}$.

In general, the process is repeated whereby the expression for the $(2n-1)$-th term serves as the incident wave for the canonical problem B, thereby producing the $2n$-th term. Then the $2n$-th term serves as the incident wave for the canonical problem C, thereby producing the $(2n+1)$-th term. This recursive process yields

$$PM((k, v, z, p); 2n+1) = [p^2 a_2^2 \eta_2(k, v) R^{-1}(k, v)] \mathscr{H}(-z) \mathscr{M}^n(k, v, h) A(k, v) J(k, v, p) \tag{16a}$$

$$\begin{aligned} PM((k, v, z, p); 2n+2) = [p^2 a_2^2(k, v) R^{-1}(k, v)] \mathscr{H}(z-h) \\ \times \mathscr{R}_p(k, v) \mathscr{H}(-h) \mathscr{M}^n(k, v, h) A(k, v) J(k, v, p) \end{aligned} \tag{16b}$$

$$n = 0, 1, 2, 3, \ldots.$$

Using (16a) and (16b), eqn (15) can be rewritten as

$$\begin{aligned} TP(k, v, z, p) &= [p^2 a_2^2 \eta_2(k, v) R^{-1}(k, v)] \\ &\quad \times \{\mathscr{H}(-z) + \mathscr{H}(z-h) \mathscr{R}_p(k, v) \mathscr{H}(-h)\} \sum_{n=0}^{\infty} \mathscr{M}^n(k, v, h) A(k, v) J(k, v, p). \end{aligned} \tag{17}$$

In eqn (17), the first term ($n = 0$) represents the waves arising from the impact loading and also the waves that have reflected once from the surface $z = h$. In general, the summation index n identifies the waves which have reflected n times from both bounding surfaces and also those which have reflected n times from the surface $z = 0$ and $n + 1$ times from the surface $z = h$. Consequently, eqn (17) contains all of the information required for the plate problem.

The convergence of the infinite matrix sum may be easily established by recalling that p has been assumed to be a real positive parameter. Moreover, p may be selected greater than any given number, this implies that X and Y may be made smaller than any given number. Since the norm of the matrix $\mathcal{M}$ consists of products of X and Y with suitable coefficients, it follows that this norm is bounded by one. Hence, by a simple application of the theory of linear bounded operators,† it follows that the infinite matrix sum in (17) converges. In fact,

$$\sum_{n=0}^{\infty} \mathcal{M}^n = (I - \mathcal{M})^{-1},$$

where I is the identity matrix and the right side of the equation is the matrix inverse of $I - \mathcal{M}$. This process leads to the spectral representation of the solution which would have been obtained by solving for the eight unknowns in eqn (9); but, as was argued in [16] for the two-dimensional case, this representation cannot be readily inverted. Consequently, to deduce an invertible representation, for $n \geq 3$, one will express the powers of the matrix $\mathcal{M}$ in the form

$$\mathcal{M}^n(k, v, h) = a_n\mathcal{M}^2(k, v, h) + b_n\mathcal{M}(k, v, h) + c_nI \tag{18}$$

which is obtained by the Cayley–Hamilton theorem. For $n = 3$,

$$a_3 = \alpha + Y^2, \qquad b_3 = Y^2(\alpha + X^2), \qquad c_3 = X^2Y^4,$$
$$\alpha = 2XY + (X - Y)^2R(k, v)/S(k, v)$$

and (18) is then the characteristic equation for $\mathcal{M}$ for the three-dimensional case.‡ For $n > 3$, eqn (18) implies the recursive relations

$$a_{n+1} = a_3a_n + b_n,$$
$$b_{n+1} = b_3a_n + c_n,$$
$$c_{n+1} = c_3a_n,$$

from which the coefficients of eqn (18) may be found. Alternatively, since the recursive relations may be combined into the form

$$a_{n+3} = a_3a_{n+2} + b_3a_{n+1} + c_3a_n$$

the general value of these coefficients may be deduced by a simple application of the theory of difference equations[21].

By the definition of $TP(k, v, z, p)$, and eqn (17), the quantities $\bar{\varphi}^{LL}$, $\bar{\psi}_x^{LL}$, and $\bar{\psi}_z^{LL}$ have been deduced. The quantity $\bar{\psi}_y^{LL}$ may be deduced from the transform of the third of eqns (1) and is given by

$$\bar{\psi}_y^{LL}(k, v, z, p) = \left(\frac{\partial\bar{\psi}_z^{LL}(k, v, z, p)}{\partial z} - pk\bar{\psi}_x^{LL}(k, v, z, p)\right)\Big/ pv. \tag{19}$$

The transforms of the displacements and stresses follow from eqns (2), (3) (17) and (19).

†For an elementary treatment of this theory see [20].

‡The two-dimensional results in [16] may be deduced from the present equations by assuming the load to be independent of y and inverting the Laplace transform on y. For example, α will reduce to the α defined in (19) of [16].

Sample problem

To effect the final inversion, the loading functions, $H(k, v, p)$, $I(k, v, p)$, and $G(k, v, p)$ must first be specified. Suppose, as an example, that a uniform normal load is suddenly applied, at time $t = 0$, for $0 \leq x, y < \infty$ as shown in Fig. 4 of [10]. Then equation (4a) becomes

$$\begin{aligned} \tau_{yz}(x, y, 0, t) &= \tau_{xz}(x, y, 0, t) = 0 \\ \tau_{zz}(x, y, 0, t) &= \mu g(x, y, t) = -\mu\delta(t)H(x)H(y). \end{aligned} \tag{20}$$

As was done in [10], this loading function is considered as the limit $\xi \to 0$ of the expression $-\delta(t)H(x)H(y)\,e^{-\xi(x+y)}$, where the limit will be taken at an appropriate step in the solution. The application of the transforms to this expression gives

$$G(k, v, p) = -\frac{1}{kp-\xi}\cdot\frac{1}{vp-\xi}, \qquad -\infty < Re(k, v) < \xi/p \tag{21}$$

with the indicated strip of convergence. This means that ϵ in eqn (7b) may be selected as zero. To illustrate the form of the contributions, one now proceeds to find the velocities in the interior of the plate. The details of the inversion of the two-sided Laplace transform on x and y are similar to those in [10, 16], and therefore only an outline of these details will be given here.

From the expression for the velocities, one can find the most general term with exponent $-p[\eta_1(k, v)w_1 + \eta_2(k, v)w_2]$:

$$\bar{f}^{LL}(k, v, z, p) = \mathscr{F}^*(k, v, 1)G(k, v, p)\,e^{-p[\eta_1(k,v)w_1+\eta_2(k,v)w_2]} \tag{22}$$

where $\mathscr{F}^*(k, v, 1)$ is the general form of this part of integrand, and w_1, w_2 depend on z and h, and are of the form $[(h-z)+nh]$, where n is an integer. Equations (22) and (7b) lead to the inversion integral

$$\bar{f}(x, y, z, p) = \frac{-1}{(2\pi)^2}\int^{i\infty}\int_{-i\infty}\frac{\mathscr{F}^*(k, v, 1)}{(k-p^{-1}\xi)(v-p^{-1}\xi)}\,e^{-p[\eta_1(k,v)w_1+\eta_2(k,v)w_2+(kx+vy)]}\,dk\,dv. \tag{23}$$

From the expression for the velocities, eqn (23) yields three distinct cases: (1) w_1 and w_2 both are nonzero; (2) w_2 is identically zero, but w_1 is not; (3) w_1 is identically zero, but w_2 is not. The evaluation of the velocities requires the analysis of these three cases. Fortunately, the inversion details for cases (2) and (3) have already been outlined in [10], and one needs only to consider case (1). For this case consider the successive transformations: (a) $k^* = -ik$, $v^* = -iv$, (b) $rk^* = \omega x - qy$, $rv^* = \omega y + qx$, $r^2 = x^2 + y^2$, and (c) $\omega = -i\sigma$. Under these transformations, eqn (23) becomes.

$$\bar{f}(x, y, z, p) = \frac{r^2}{i(2\pi)^2}\int_{-\infty}^{\infty}dq\int_{-i\infty}^{i\infty}\frac{\mathscr{F}(-i\sigma, q, 1)\,e^{-pt}\,d\sigma}{(\sigma x - iqy - r\xi/p)(\sigma y + iqx - r\xi/p)} \tag{24}$$

where t is defined by

$$t = \eta_1(-i\sigma, q)w_1 + \eta_2(-i\sigma, q)w_2 + r\sigma, \tag{25}$$

and $\mathscr{F}$ is obtained from $\mathscr{F}^*$ by the transformations. Referring to Fig. 5 of [10], the singularities of the integrand in the σ-plane are the branch points at $\pm(a_1^2 + q^2)^{1/2}$, $\pm(a_2^2 + q^2)^{1/2}$, and the simple poles σ_1, σ_2, and $\pm(q^2 + c_R^{-2})^{1/2}$, where c_R is the Rayleigh wave speed, and

$$\sigma_1 = (-iqx + r\xi/p)y, \qquad \sigma_2 = (iqy + r\xi/p)/x.$$

Since r is a positive quantity, the contour of eqn (24), as referred to Fig. 5 of [10], may be closed only to the right of the imaginary axis. The pole σ_1 (or σ_2) lies inside the contour only for y (or x) positive, and the residue theorem may be applied to the integral. As shown in [21], the contributions resulting from the poles σ_1 and σ_2 may be evaluated by the techniques in [10] and

[16]. Consequently, to indicate the features required for the three-dimensional case which are different from those of [10] and [16], it will be assumed that both x and y are negative.

Exact inversion

By the branch cut selection, $\eta_\alpha(-i\sigma, q)$ for $\alpha = 1, 2$ is a purely imaginary quantity to the right of $(a_\alpha{}^2+q^2)^{1/2}$ on the real axis of the σ-plane, and is a real quantity on the real axis between the branch points and also on the imaginary σ-axis. Following [10, 16], let t be a real parameter generating the Cagniard path defined by $t = (a_1{}^2+q^2-\sigma^2)^{1/2}w_1+(a_2{}^2+q^2-\sigma^2)^{1/2}w_2+r\sigma$. Since t is a real quantity, this implies that the intercept of the path with the real σ-axis is located between the origin and $(a_1{}^2+q^2)^{1/2}$; thus, there is no branch cut contribution and the Cagniard contour V in Fig. 5 of [10] is the required path. Equation (24) reduces to

$$\tilde{f}(x, y, z, p) = \frac{r^2}{i(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}q \int_{\Gamma_5} \frac{\mathcal{F}(-i\sigma, q, 1)\,\mathrm{e}^{-pt}\,\mathrm{d}\sigma}{(\sigma x - iqy - r\xi/p)(\sigma y + iqx - r\xi/p)}, \tag{26}$$

where Γ_5 is the Cagniard contour V of [10]. One now takes the limit as ξ goes to zero. Recalling that q is a real variable, one immediately concludes that there are no singularities for σ along the Cagniard path. The values of σ along the Cagniard path are obtained by prescribing the real quantities t and q and solving the equation

$$t = (a_1{}^2+q^2-\sigma^2)^{1/2}w_1+(a_2{}^2+q^2-\sigma^2)^{1/2}w_2+\sigma r. \tag{27}$$

for σ in terms of t, q, w_1, w_2, and r. The arguments presented in Appendix B of [16] are directly applicable here, provided that the following substitutions be made in eqn (B1): $k = \sigma$, $x = r$, $a_1{}^2 = a_1{}^2+q^2$, $a_2{}^2 = a_2{}^2+q^2$, $w = w_1$, $z = w_2$. Thus, one finds that on Γ_5 the inequality

$$t \geq (a_1{}^2+q^2)^{1/2}w_1+(a_2{}^2+q^2)^{1/2}w_2$$

holds. Also, the intercept of the Cagniard path with the real σ-axis, denoted by σ_0, is the solution of $\partial t/\partial\sigma = 0$; that is, by eqn (27),

$$\left.\frac{\partial t}{\partial \sigma}\right|_{\sigma_0} = [r - \sigma w_1/\eta_1(-i\sigma, q) - \sigma w_2/\eta_2(-i\sigma, q)]|_{\sigma=\sigma_0} = 0, \tag{28}$$

implying that σ_0 is a function of q^2. The value of t at this point is obtained by substituting σ_0 into eqn (27) to obtain

$$t_{\text{int}} = g(q^2), \tag{29}$$

where g is a certain function of q^2. The case when w_1w_2 is nonzero precludes the explicit determination of the Cagniard path needed for the solution. However, it is of importance, for the subsequent change of the order of integration, to note that the value t at the intercept is an even function of q.

Denote by σ_c the root of eqn (27) in the first quadrant of the σ-plane, and note that σ_c is a function of q^2. It is easy to show that $\bar{\sigma}_c$, the complex conjugate of σ_c, is also on the Cagniard path. Hence, the contribution from the first quadrant of the σ-plane may be written as

$$\tilde{I}(x, y, z, p) = \frac{r^2}{i(2\pi)^2}\int_{-\infty}^{\infty}\int_{t_{\text{int}}}^{\infty} \frac{\mathcal{F}(-i\sigma_c, q, 1)}{(\sigma_c x - iqy)(\sigma_c y + iqx)}\frac{\partial \sigma_c}{\partial t}\mathrm{e}^{-pt}\,\mathrm{d}q\,\mathrm{d}t. \tag{30}$$

This double integral may be put into a more convenient form by interchanging the order of integration, as shown in Fig. 6 of [10], with the substitutions in the Figure of $g(0)$ and $t = g(q^2)$ instead of a_1R and $t = R(a_1{}^2+q^2)^{1/2}$, obtain

$$\tilde{I}(x, y, z, p) = \frac{r^2}{i(2\pi)^2}\int_{g(0)}^{\infty} \mathrm{e}^{-pt}\,\mathrm{d}t \int_{-f_1(t)}^{+f_1(t)} \frac{\mathcal{F}(-i\sigma_c, q, 1)}{(\sigma_c x - iqy)(\sigma_c y + iqx)}\frac{\partial \sigma_c}{\partial t}\mathrm{d}q$$

where $f_1(t)$ is the inverse function obtained by solving for q from $t = g(q^2)$. Since the limits of the inside integral are $\pm f_1(t)$, only the part of the integrand which is an even function of q will contribute to the solution. Thus, it seems appropriate to define $\mathscr{E}(\sigma_c, q, t)$ as the part of the integrand which is an even function of q, and then it follows from an application of Cagniard's technique that

$$\begin{aligned}\bar{I}(x, y, z, p) &= \frac{r^2}{i(2\pi)^2}\int_0^\infty H[t-g(0)]\,\mathrm{e}^{-pt}\,\mathrm{d}t\int_0^{f_1(t)} 2\mathscr{E}(\sigma_c, q, t)\,\mathrm{d}q \\ I(x, y, z, t) &= \frac{2r^2}{i(2\pi)^2}H[t-g(0)]\int_0^{f_1(t)} \mathscr{E}(\sigma_c, q, t)\,\mathrm{d}q.\end{aligned} \tag{31}$$

For definiteness, the evaluation of this equation will now be indicated for given values of r, w_1, w_2, and t. First one sets q equal to zero in eqn (28) and solves for σ_0. Using this value σ_0 and setting q equal to zero in eqn (27), one finds $t_{\text{int}} = g(0)$. The finding of $f_1(t)$ is equivalent to finding the saddle point which lies on the path of steepest descent Γ_5 (i.e. the Cagniard path is a path of steepest descent). This requires solving simultaneously eqns (27) and (28) for q and σ_0. The solution q^* is then the desired value of the upper limit in (31); that is, $f_1(t) = q^*$. For the integrand, q is also given such that $0 \leq q \leq f_1(t)$, and σ_c is found from eqn (27). The evaluation of $\mathscr{E}(\sigma_c, q, t)$ and of the whole eqn (31) follows from these arguments.

The contribution from the fourth quadrant of the σ-plane may be obtained in a similar manner by beginning with eqn (30) where the inner integral has the limits interchanged and $\bar{\sigma}_c$ is used instead of σ_c. Thus, eqns (22)–(31), and the accompanying discussion, illustrate the details of the analysis required for the three-dimensional case which are different from those of Ref. [10, 16, 22]. These details represent a generalization of the techniques employed in these references.

Numerical results

To complement the preceding theoretical development, the response to a step time input will now be determined. In this case, the boundary condition for τ_{zz} becomes

$$\tau_{zz}(x, y, 0, t) = \mu g(x, y, t) = -\mu H(t)H(x)H(y).$$

The velocity for this input is obtained from the velocity $\mathbf{v}^\delta$ resulting from the load given by eqn (20). This velocity is given by

$$\mathbf{v}(x, y, z, t) = \int_0^t \mathbf{v}^\delta(x, y, z, \tau)H(t-\tau)\,\mathrm{d}\tau.$$

For example, the contribution from the term given by eqn (31) will be

$$L(x, y, z, t) = \frac{2r^2}{i(2\pi)^2}\int_0^t \left\{H[\tau - g(0)]\int_0^{f_1(\tau)} \mathscr{E}(\sigma_c, q, \tau)\,\mathrm{d}q\right\}H(t-\tau)\,\mathrm{d}\tau,$$

which reduces to

$$L(x, y, z, t) = \frac{2r^2}{i(2\pi)^2}H[t-g(0)]\int_{g(0)}^t\int_0^{f_1(\tau)} \mathscr{E}(\sigma_c, q, \tau)\,\mathrm{d}q\,\mathrm{d}\tau.$$

For computational convenience, in the numerical results of this problem, one introduces dimensionless variables with c_1, c_1/h, and h as the rationalizing quantities. A computer program was written for the velocity vector which encompassed all the mathematical difficulties which arose in the problem. The actual integration was performed using a CDC-6600 computer. To illustrate the results obtained, two plots for the point $x = 0.82h$, $y = 0.41h$, and $z = 0.8h$ will now be presented for Poisson's ratio of 0.25. In both plots, the dimensionless time variable $T = tc_1/h$ and the dimensionless velocity $V_z = V_z/c_1$ are given. In Fig. 2, the z-velocity without reflections

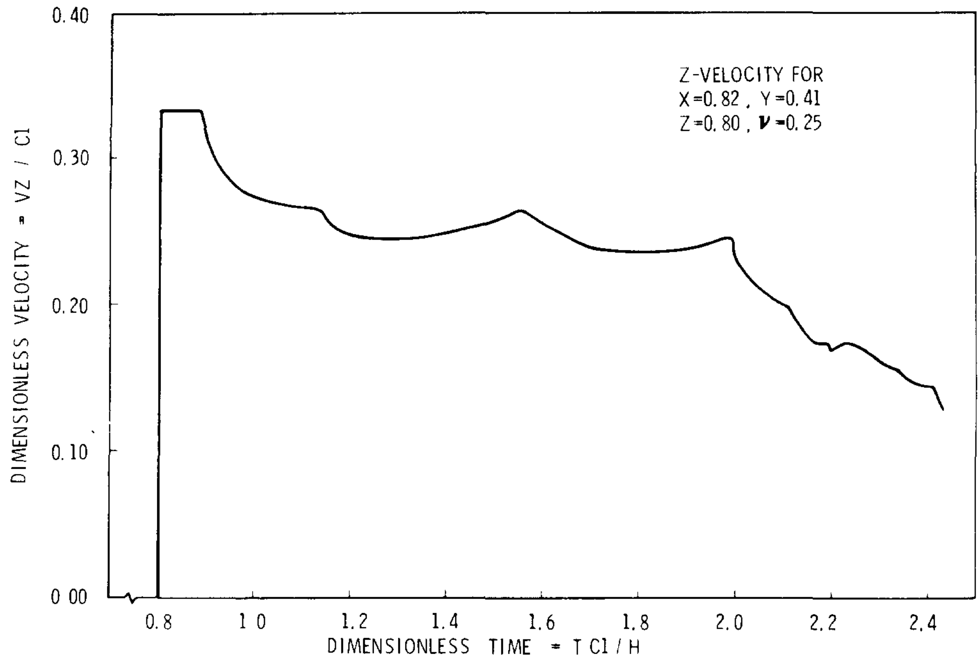

Fig. 2. V_z for the elastic half-space.

(in an elastic half-space) is shown. In Fig. 3, the z-velocity with reflections (in an elastic thick plate) is shown. In both figures, one sees the jump in the z-velocity at the first wave arrival. The second wave is a release wave which reduces the z-velocity induced by the first wave. Another jump in Fig. 3 occurs at the arrival of the fourth wave, which is a reflected plane wave. In each case, the value of the jump is 1/3. For Fig. 3, the computation was performed up to a time larger than the arrival of a wave for which both w_1 and w_2 are non-zero. Hence, for longer times, one merely needs to change the values of w_1 and w_2 in the appropriate places, and the evaluation will proceed with no further complications. Thus, the power and uses of the Ray-Grouping technique have been demonstrated, and one can easily verify that all the required details for evaluating the analytic solution involve no further difficulties than those already considered.

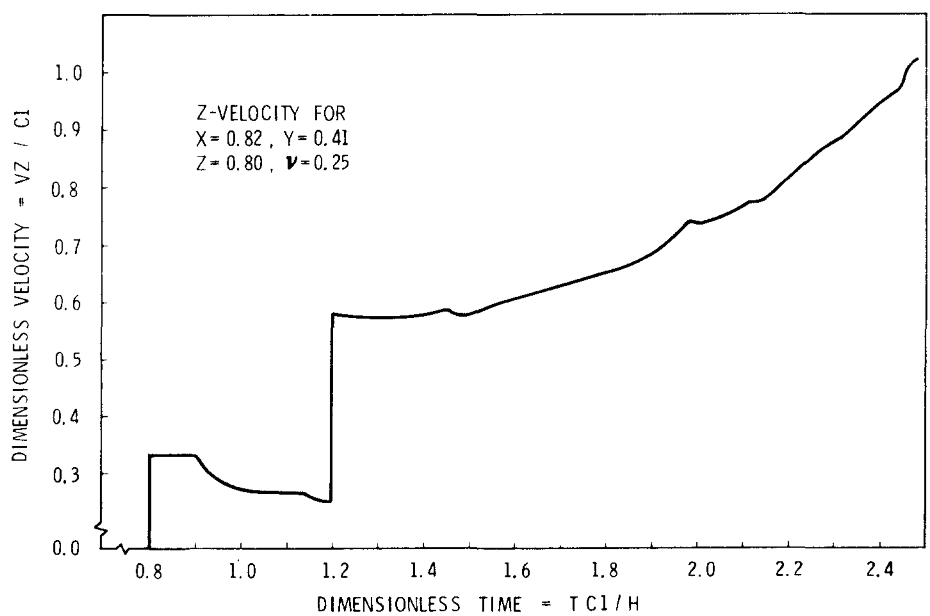

Fig. 3. V_z for the plate geometry.

For the load selected, the analytical solution contains the solution to the problems posed in Ref. [10] and [16]. The solution for the problem in Ref. [10] is obtained by neglecting the reflected waves and numerical results for this case appear in Fig. 2. For the solution to the problem in Ref. [16], with a suitable change in the coordinate labels, one can easily identify the contributing terms from the analytical expressions.† For the numerical program, this may be accomplished by selecting a large value for y. Thus, the program used for generating the preceding two figures was evaluated at the point $x = 0.5h$, $z = 0.5h$, and $y = 200h$. The resulting time history did not agree with the results of [16]. but a careful analysis revealed a sign mistake in the program used in [16]. While the present analytical work was in progress, D. L. Hicks *et al.*, were developing Threedy, a three-dimensional wave propagation code which employs operator splitting[23] and includes non-linear material laws. Using the notation of [16], Fig. 4 shows the velocity history computed from the analytical solution and compares it to the numerical results generated by Hicks *et al.*‡ Recalling tha one of the objectives of [16] was to provide exact results for numerical code development, the agreement is quite good. Figure 4 is presented here as a replacement for Fig. 4 in [16].

DISCUSSION

This paper considered the problem of an elastic thick plate to which an arbitrary dynamic loading was applied on one of its bounding surfaces, while the other bounding surface was assumed stress free. As was done in [16], the motion in the interior of the plate was found by the application of one-sided and two-sided Laplace transforms, and by the use of the Ray Grouping technique. This technique required the solution of three canonical problems whose solution was used to generate the solution to the plate problem. In addition, this technique provides a physical interpretation of each term in the solution. By incorporating the use and results of matrix theory, the Ray Grouping technique tremendously simplifies the calculations required by the ray tracing method, so that $(m+n)!/m!n!$ terms appearing in the ray tracing method result in only one term in the Ray Grouping technique. The technique may be extended to include different boundary conditions at the stress free surface of the plate, including possible contact with a fluid and another elastic material. Hence, this technique is readily

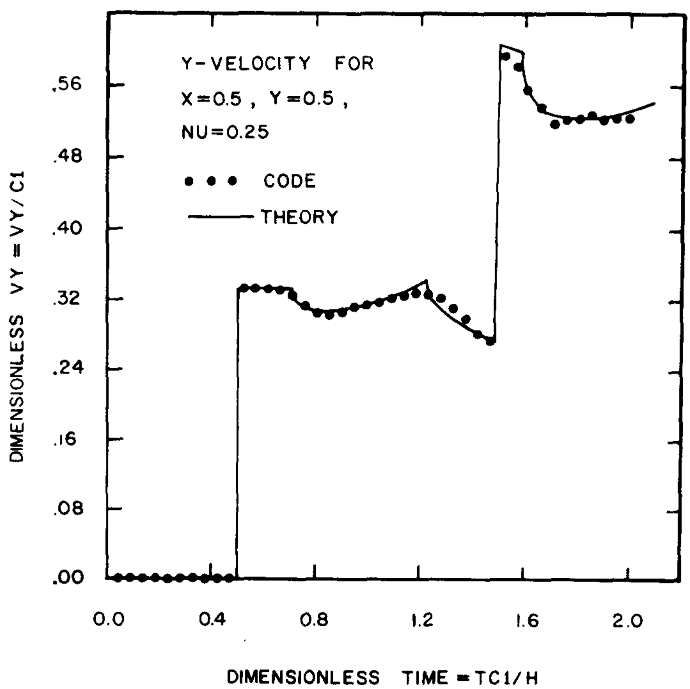

Fig. 4. V_y for the two-dimensional case.

†As was done in eqn (47) of [10].

‡This is preliminary to a more detailed comparison, to be published elsewhere, of Threedy and the analytical solution to some two-dimensional codes.

applicable to the case of an elastic layered medium. For the case of one layer over a half space, the only change will be to replace boundary conditions (4b) with some suitable conditions at the interface, such as continuity of stress and/or displacement across the interface.

To demonstrate the technique, the non-axially symmetric problem of a uniform normal load suddenly applied on a quarter surface of the plate was considered. The details of the transform inversion required for the plate problem, which are different from those of Refs. [10, 16, 22], were carefully outlined. These details represent a generalization of the techniques employed in these references. The superposition of the solution for a uniform load acting on a half-infinite strip of finite width and on a finite rectangular region may be accomplished by the procedure indicated in [10].

It is hoped that the present work will serve as a contribution to the theoretical studies of elastic plates and layered elastic half-spaces. The techniques and results developed here are quite general, and their extension to the layered case will be presented in the near future. Thus, it is easily seen that the convergence established here, plus the details of the transform inversion will be directly applicable to the layered case, so that these two items need not be emphasized in the solution of the layered case.

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